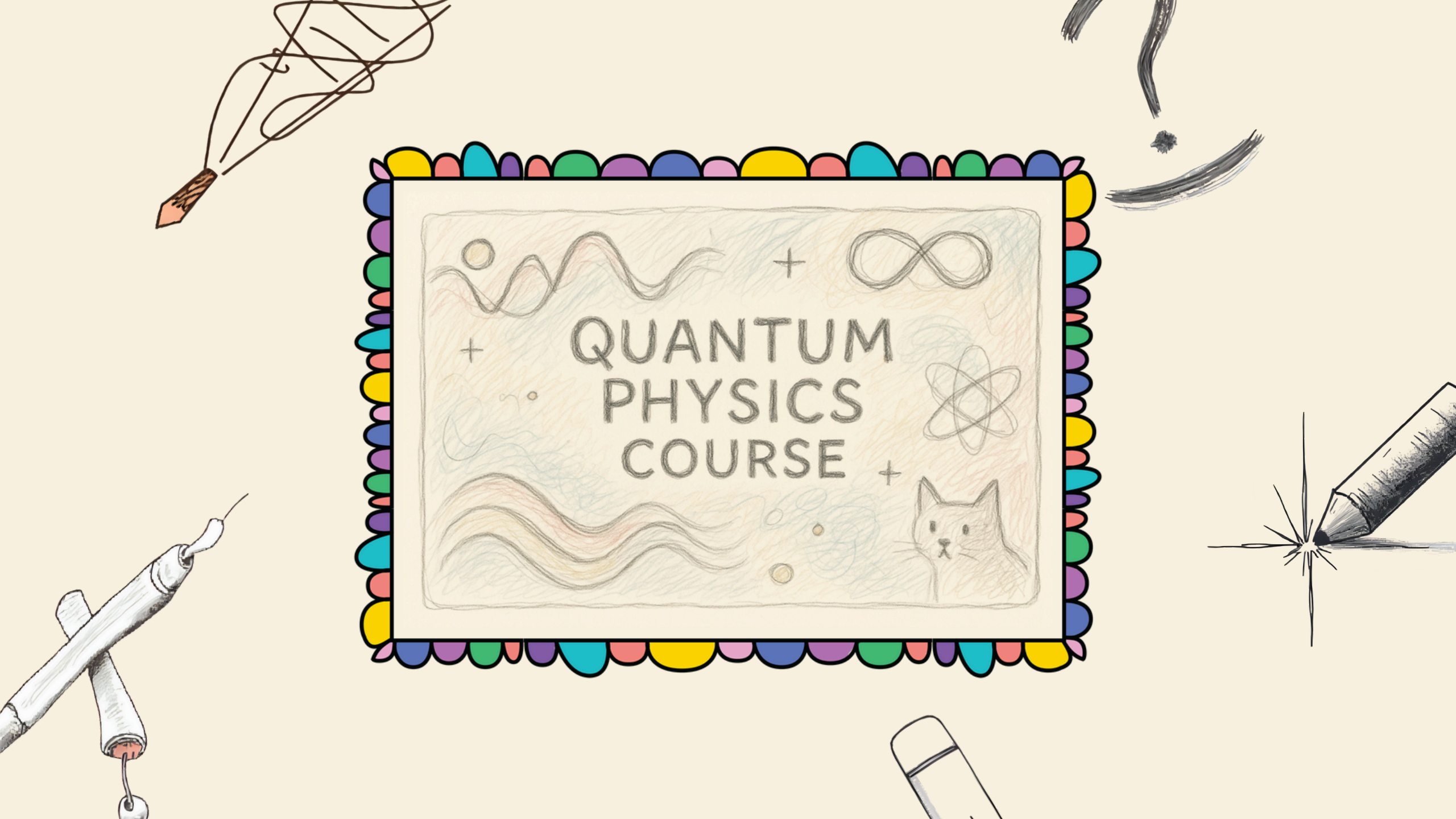

QUANTUM
PHYSICS
COURSE

So, a fine way to start is harmonic oscillator !

$f = ma$ (newton's formula)

$f = m \frac{d^2}{dt^2} \underbrace{u(x+h,t)}$ { more detailed }

what's this ??

↳ well,

$u \longrightarrow$ displacement

$(x+h, t) \longrightarrow$ factors on which u depends

you know Δx

{ $F = ma = -kx$ }

- imagine a system of spring & particles :-

$F_{net} - F_x = k\left(u(x+2h,t) - u(x+h,t)\right) - k\left(u(x+h,t) + u(x,t)\right)$

$F_{net} = k\left(u(x+2h,t) - u(x+h,t)\right) - k\left(u(x+h,t) + u(x,t)\right)$

{ $F = ma(t) = m \frac{\partial^2 (u(x+h,t))}{\partial t^2}$ }

substitute the value of F_{net} into it :

$\frac{\partial^2}{\partial t^2} u(x+h,t) = \frac{k}{m}\left[u(x+2h,t) - u(x+h,t)\right] - \frac{k}{m}\left[u(x+h,t) - u(x,t)\right]$

$\frac{\partial^2 u(x+h,t)}{\partial t^2} = \frac{k}{m} u(x+2h,t) - \frac{k}{m} u(x+h,t) - \frac{k}{m} u(x+h,t) + \frac{k}{m} u(x,t)$

$\frac{\partial^2 u(x+h,t)}{\partial t^2} = \frac{k}{m} u(x+2h,t) - \frac{2k}{m} u(x+h,t) + \frac{k}{m} u(x,t)$

now, as there are multiple masses,

let there be N masses

$\longrightarrow L$ (Total length of the system) $= Nh$ ↳ { h }

$\longrightarrow k = k/N$

$\longrightarrow M = Nm$ ↳ sum of single particle

↳ final system's mass

(∴ system has identical mass containing particles)

hence, the final equation shall be :

$$\frac{\partial^2}{\partial t^2} u(x+h,t) = \frac{KL^2}{M} \frac{u(x+2h,t) - 2u(x+h,t) + u(x,t)}{h^2}$$

- now, a little understanding

$f'(x) = \lim_{x \to 0} \frac{f(x+h) - f(x)}{h}$

$f''(x) = \lim_{x \to 0} \frac{f'(x+h) - f'(x)}{h}$

$\Rightarrow \lim_{x \to 0} \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2}$

the part { } is a second order central difference approximation of the second order derivative : $\frac{\partial^2}{\partial x^2} u(x+h,t)$

now, in here, when $N \to \infty$ and $h \to 0$ then,

the whole

{ $\frac{f(x+2h) - 2f(x+h) + f(x)}{h^2} = \frac{\partial^2 u(x+h,t)}{\partial x^2}$ }

↳ $\frac{\partial^2}{\partial t^2} u(x,t) = \left(\frac{KL^2}{M}\right) \frac{\partial^2}{\partial x^2} u(x,t)$

↳ c^2

hence

$$\frac{\partial^2}{\partial t^2} u(x,t) = c^2 \frac{\partial^2}{\partial x^2} u(x,t)$$

↳ wave equation !!

btw if by any chance you've wondering that how come $c^2 = \frac{KL^2}{M}$

then feel free to have a dimensional check !! 🐭

Hey !! have you just noticed what we've done?! we defined the wave equation from the quantised fundamental system !!

also we do have one more way to derive this beautiful equation !

→ to be continued ...

now, it begins from a very basic form of travelling waves :-

$\psi(x,t) = f(x-ct) + h(x+ct)$

as you can see it's pretty due to the form it's in, consisting of every possibility in waves ...

$\psi(x,t) = \underbrace{f(x-ct)}_{\text{part of equation portraying wave propagating in +ve dire}} + \underbrace{h(x+ct)}_{\text{part of equation presenting wave propagating in -ve dire}}$

so, let a wave to be :-

$\psi(x,t) = Af(x-vt)$

now, let first differentiate $\psi(x,t)$ w.r.t x

$$\Rightarrow \frac{\partial \psi(x,t)}{\partial x} = A \frac{\partial f(x-vt)}{\partial (x-vt)} \cdot \frac{\partial (x-vt)}{\partial x}$$

can write it as, $\frac{\partial \psi}{\partial x} = Af'$

$\longrightarrow$ differentiate $\frac{\partial \psi}{\partial x}$ w.r.t x

$$\Rightarrow \frac{\partial f'(x-vt)}{\partial (x-vt)} \cdot \frac{\partial (x-vt)}{\partial x}$$

$$\Rightarrow \frac{\partial^2 \psi}{\partial x^2} = Af''$$

$\longrightarrow$ now, differentiate $\psi(x,t)$ w.r.t t

$$\frac{\partial \psi(x,t)}{\partial t} = A \frac{\partial f'(x-vt)}{\partial (x-vt)} \cdot \frac{\partial (x-vt)}{\partial t}$$

$$\Rightarrow \frac{\partial \psi}{\partial t} = -vAf'$$

$\longrightarrow$ differentiate $\frac{\partial \psi}{\partial t}$ w.r.t t :-

$$\frac{\partial^2 \psi}{\partial t^2} = -vA \frac{\partial f'(x-vt)}{\partial (x-vt)} \cdot \frac{\partial (x-vt)}{\partial t}$$

$$\Rightarrow \frac{\partial^2 \psi}{\partial t^2} = v^2 A f''$$

now, from the above equations substitute (Af'') we'll get

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \frac{\partial^2 \psi}{\partial x^2}$$

$$\frac{\partial^2 \psi}{\partial t^2} = c^2 \frac{\partial^2 \psi}{\partial x^2}$$ — wave equation

now, we'll head towards the solution part of the wave equation !

so as you can tell,

$\psi(x,t) = X(x) \cdot T(t)$

put it into the wave equation :-

$$\frac{\partial^2 (X(x)\,T(t))}{\partial t^2} = c^2 \cdot \frac{\partial^2 (X(x)\,T(t))}{\partial x^2}$$

$$\Rightarrow X(x) \cdot \frac{\partial^2 (T(t))}{\partial t^2} = c^2\, T(t) \cdot \frac{\partial^2 (X(x))}{\partial x^2}$$

$$\Rightarrow \underbrace{\frac{1}{T(t)} \frac{\partial^2 T(t)}{\partial t^2}}_{\text{depends on } t} \quad \underbrace{c^2 \frac{1}{X(x)} \cdot \frac{\partial^2 X(x)}{\partial x^2}}_{\text{depends on } x}$$

as of now we can see :-

they must bot equal the same constant (say $-\lambda$)

now i know, that you know but

λ isn't wavelength — still !

λ is a constant related to the wavelength, which is related to the shape of the respected wave

back to where we were !!

we'll get,

$$\frac{1}{T} \frac{\partial^2 T}{\partial t^2} = -\lambda \quad ; \quad \frac{1}{X} \frac{\partial^2 X}{\partial x^2} = \frac{-\lambda}{c^2}$$

general solution of these ODEs :-

time part :

$T(t) = A\cos(\omega t) + B\sin(\omega t)$; $\omega = \sqrt{\lambda}$

space part :

$X(x) = C\cos(kx) + D\sin(kx)$; $k = \frac{-\sqrt{\lambda}}{c}$

$\psi(x,t) = X(x) \cdot T(t)$

$\longrightarrow$ to be continued

$$\Downarrow$$

$$\psi(x,t) = (A\cos(\omega t) + B\sin(\omega t)) \cdot (C\cos(kx) + D\sin(kx))$$

which apparently also be written as :-

$$\psi(x,t) = f(x-ct) + g(x+ct)$$

btw so that you won't get confused remember

→ wavelength = λ_{phy}

& ↳ $\left\{ \lambda_{phy} = \frac{2\pi}{k} = \frac{2\pi c}{\sqrt{\lambda}} \right\}$

• also you can represent the same in exponential form as :-

$$\psi(x,t) = (Ae^{i\omega t} + Be^{-i\omega t}) \cdot (Ce^{ikx} + De^{-ikx})$$

well i should assume that you're totally aware of the entities featured in the :-

$$\psi(x,t) = A\sin(\omega t \pm kx)$$

but let me still clear 'em for ya !

→ to be continued

So,

$\psi(x,t) = A\sin(\omega t \pm kx)$

- A → Amplitude
- ω → angular frequency
- k → wavenumber ($k = \frac{1}{\lambda}$)
- $\omega = \frac{2\pi}{T} = 2\pi\nu$, ν → frequency ($\nu = \frac{1}{T}$)

$v = \frac{-\omega}{k}$ } if v has +ve sign ⇒ wave will travel in forward diren

If v has −ve sign ⇒ wave will travel in backward diren

(Here 'v' is just context-dependent term used to show diren of propagation)

example

$\psi(x,t) = A\sin(\omega t - kx)$

example

$\psi(x,t) = A\sin(\omega t + kx)$

• Superposition of waves :-

let there be ~~2 waves~~ Hyperx! Kidding!! two waves*

$\psi_1 = \psi_0 \sin(kx - \omega t)$

$\psi_2 = \psi_0 \sin(kx - \omega t + \phi)$

now, the superposition shall be ...

$\psi_s = \psi_1 + \psi_2$

$\psi_s = \psi_0 \sin(kx - \omega t) + \psi_0 \sin(kx - \omega t + \phi)$

$\psi_s = 2\psi_0 \cos\left(\frac{\phi}{2}\right) \sin\left(kx - \omega t + \frac{\phi}{2}\right)$

value of ψ_s :-

interference { max when $\phi \longrightarrow 0$; min when $\phi \longrightarrow \pi$

as you do know,

$e^{i\theta} = \cos\theta + i\sin\theta$

$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

• also for different frequencies, superposition would be :-

$\psi(x,t) = \psi_0 \sin(k_1 x - \omega_1 t) + \psi_0 \sin(k_2 x - \omega_2 t)$

$= 2\psi_0 \cos\left[\left(\frac{k_1 - k_2}{2}\right)x - \left(\frac{\omega_1 - \omega_2}{2}\right)t\right] \times \sin\left[\left(\frac{k_1 + k_2}{2}\right)x - \left(\frac{\omega_1 + \omega_2}{2}\right)t\right]$

$\frac{\omega_1 - \omega_2}{2}$ } beat frequency

$\frac{\omega_1 + \omega_2}{2}$ } sound frequency

well, if you'd like, then, to explore it's relation/application into Quantum mech. in the generation of Quantum beats — you can refer to — The Feynman Lectures on Physics Vol. 1 Ch. 48

let's get onto some energetics :-

$\psi(x,t) = a\cos(\omega t - kx)$

Kinetic energy $= \frac{1}{2}mv^2$ & $\left(v = \frac{\partial\psi}{\partial t}\right)$

{ a little sweet declaimer : whatever i am about to show you here at the part of energetics are not necessarily be concerning for every kind of wave ; precisely not for the kind of quantum scale waves heading to ! — this is for more of a classical type of waves (eg. String) — cuzz energetics for Quantum type are certainly defined via Hamiltonian operation followed with energy expectation value ... so so }

continued :-

$(m = \rho\delta x)$ $\left(\frac{\partial\psi}{\partial x} = \frac{\partial(a\cos(\omega t - kx))}{\partial x}\right)$

$-a\sin(\omega t - kx)\,\omega$

$-a\omega\sin(\omega t - kx)$

therefore,

$KE = \frac{1}{2}mv^2$

$\Rightarrow \frac{1}{2}\rho\delta x\left(\frac{\partial\psi}{\partial t}\right)^2$

$\Rightarrow \frac{1}{2}\rho\delta x\, a^2\omega^2\sin^2(\omega t - kx)$

KE per unit length : $\frac{1}{2}\rho\delta a^2\omega^2\sin^2(\omega t - kx)$

→ to be continued

KE per unit length of a transverse motion is actually just Kinetic energy density, taking a section (δx) having mass $(\mu_0 \delta x)$ & $\left(v = \frac{\partial y}{\partial t}\right)$.

now, Potential energy :-

think of a string lies along x-axis at equilibrium.

→ at time t, string is displaced to a new shape given by $y(x,t)$ } vertical displacement

→ tiny segment $(x+\delta x)$

original length → δx
new length after displacement ⟶ bw two points
$A\,(x, y(x))$
$B\,(x+\delta x, y(x+\delta x))$

use distance formula :

$$\delta l = \sqrt{(x+\delta x - x)^2 + (y(x+\delta x) - y(x))^2}$$

$$\delta l = \sqrt{(\delta x)^2 - (\delta y)^2} \quad \text{where,}$$

$$\{\delta y = y(x+\delta x) - y(x)\}$$

So,

$$\delta l = \sqrt{(\delta x)^2 + (y(x+\delta x) - y(x))^2}$$

⇒ divide numerator & denominator by (δx^2)

$$\delta l = \delta x \sqrt{1 + \left(\frac{y(x+\delta x) - y(x)}{\delta x}\right)^2}$$

and as $\delta x \to 0$, the fraction becomes partial derivative $\left(\frac{\partial y}{\partial x}\right)$:-

$$\delta l = \delta x \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2}$$

why this worth you attention ? cuzzz....

a little stretch $(\delta l - \delta x)$ stores energy like a spring, and the difference tells us how much potential energy is stored.

Potential energy density : $\frac{\text{Work done to stretch}}{\text{segment length}}$

$$= \frac{T(\delta l - \delta x)}{\delta x} \qquad (T \text{ is tension})$$

which leads,
(via binomial approximation)

$$PE_d = \frac{1}{2} T \left(\frac{\partial y}{\partial x}\right)^2$$ } potential energy density

and hence, the total energy density would be ;

$$u(x) : \frac{1}{2}\left(\rho \left(\frac{dy}{\partial x}\right)^2 + T\left(\frac{\partial y}{\partial x}\right)^2\right)$$ energy density

to look deep into energetics I insist you to check out

users.physics.ox.ac.uk for Energy Handout PDF and I'll try to attach it here somewhere :)

after hitting up to energetics, lets jump onto the intensity part :-

Intensity ⇒ I = Energy density × Wave speed
(energy flux)

⇒ $I(x,t) = u(x,t) \cdot v$

pss. & we know $v = \sqrt{\frac{T}{\mu}}$

average intensity over time ;

$$\langle I(x,t) \rangle = \langle u(x,t) \rangle \cdot v$$

for sinusoidal wave equation :-

$$\langle I \rangle \propto A^2 v$$

~ To be continued

↳ continuing ...

so in Quantum wavefunction's context "intensity" can be loosely connected to probability current ...

$$j(x,t) = \frac{i\hbar}{2m}\left(\psi \frac{\partial \psi^*}{\partial x} - \psi^* \frac{\partial \psi}{\partial x}\right)$$

this shows probability flows, as it's which — while not energy — serves a similar purpose conceptually,

energy current in quantum field theory can be derived from energy-momentum tensors but that's not relelent to you for now...also, at the time I'm writing this " I am totally oblivious about such"

hence, have fun ^ω^

OKAY !!

you remember this ??

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \frac{\partial^2 \psi}{\partial x^2}$$

you do ??

good !!

for a free particle, let ;

$\psi = Ae^{i(kx-\omega t)}$ } general wave equation

$E_k = \frac{1}{2}mv^2 = \frac{p^2}{2m} = \hbar\omega$

$p = \hbar k$

substitute value ;

$\psi = Ae^{i(\frac{p}{\hbar}x - \frac{E}{\hbar}t)}$

$\psi = Ae^{\frac{i}{\hbar}(px - Et)}$

differentiate w.r.t (x) ;

$$\frac{\partial \psi}{\partial x} = A e^{\frac{i}{\hbar}(px-Et)} \cdot \left(\frac{i}{\hbar}p\right)$$

$$\Rightarrow \frac{\partial \psi}{\partial x} = \psi\left(\frac{i}{\hbar}p\right)$$

$$\Rightarrow -i\hbar \frac{\partial \psi}{\partial x} = p\psi$$

differentiate it again w.r.t (x) ;

$$\frac{\partial^2 \psi}{\partial x^2} = \left(\frac{i}{\hbar}p\right)\left(\frac{i}{\hbar}p\right) A e^{\frac{i}{\hbar}(px-Et)}$$

$$\Rightarrow \frac{\partial^2 \psi}{\partial x^2} = -\frac{p^2}{\hbar^2}\psi$$

now, differentiate ψ w.r.t (t) ;

$$\frac{\partial \psi}{\partial t} = A e^{\frac{i}{\hbar}(px-Et)} \cdot \left(-\frac{i}{\hbar}E\right)$$

$$\Rightarrow \frac{\partial \psi}{\partial t} = \psi\left(-\frac{i}{\hbar}E\right)$$

$$\Rightarrow i\hbar \frac{\partial \psi}{\partial t} = E\psi$$

and as $E = \frac{p^2}{2m}$

$$\rightarrow E\psi = \frac{p^2}{2m}\psi$$

substituting $(p^2\psi)$'s term from previously derived equalities ;

$$E\psi = \left(-\frac{\hbar}{2m}\frac{\partial^2 \psi}{\partial x^2} = i\hbar \frac{\partial \psi}{\partial t}\right)$$

↳ and this is time-dependent Schrödinger's equation

$E = \frac{p^2}{2m} + V(x)$

multiplying ψ both sides ;

$$\Rightarrow E\psi = \frac{-\hbar}{2m}\psi + V(x)\psi$$

$$\left(-\frac{\hbar}{2m}\cdot\frac{\partial^2 \psi}{\partial x^2} + V(x)\psi = E\psi\right)$$ } Shrödinger's Equation (time independent)

just in case ;

$\hbar = \frac{h}{2\pi}$

$h = 6.626 \times 10^{-34}$ Js

the wave function ;

$\psi(x,t) = \psi_x e^{-iEt/\hbar}$

does obviously depends on t.

$$|\psi(x,t)^2| = \psi^*\psi = (\psi_x e^{-iEt/\hbar})\cdot(\psi_x e^{iEt/\hbar}) = |\psi(x)|^2$$

the probability density does not,

the same thing happens in calculating the expectation value of any dynamical variable

↳ what is it ??

↳ will come !!

as we know that, free wavefunction, carries that exponential time-dependent factor containing ; $\langle x \rangle$ = constant, hence $\langle p \rangle = 0$

in classical mechanics, total energy (KE+PE) is called the Hamiltonian

~ to be continued

$H(x,p) = \frac{p^2}{2m} + V(x)$

Hamiltonian operator, obtained from canonical substitution;

$\left\{ p \to \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right) \right\}$

$\hat{H} = -\frac{\hbar}{2m}\frac{\partial}{\partial x^2} + V(x)$

hence, Schrödinger Eq :

$\hat{H}\Psi = E\Psi$

group of waves,

$\Psi(x,t) = \sum_{n=1}^{\infty} C_n \Psi_n(x,t)$

or,

$\Psi(x,t) = \sum_{n=1}^{\infty} C_n \Psi_n(x) e^{-i\omega_n t}$

now, from this time-dependent Schrodinger equation,

$\frac{-\hbar}{2m}\frac{d^2\Psi}{dx^2} = i\hbar\frac{d\Psi}{dt}$

a general solution can be expressed as a superposition of plane waves;

$\Psi(x,t) = \int \phi(k) e^{i(kx - \omega(k)t)} \cdot dk$

$\phi(k)$ = Amplitude of each wave component (Fourier space wave function)

$\omega(k)$ = angular frequency as a function of k

dispersion relation,

as we know $\rightarrow E = \frac{p^2}{2m} = \frac{(\hbar k)^2}{2m} = \hbar\omega$

assume;

$\omega(k) = \frac{\hbar k^2}{2m}$ } this is non-linear indicating components of wave packet (with different k) travel at diff. speeds

assume $\phi(k)$ is sharply peaked around;

$k = k_0$, then expand $\omega(k)$ about k_0 :

$\omega(k) \approx \omega(k_0) + (k-k_0)\left.\frac{d\omega}{dk}\right|_{k_0}$

substitute into wave packet :-

$\Psi(x,t) \approx e^{i(k_0 x - \omega(k_0)t)} \int \phi(k) e^{i(k-k_0)(x - \frac{\hbar k_0}{m}t)} \cdot dk$ ← group velocity

$; v_g = \left.\frac{d\omega}{dk}\right|_{k_0} \Rightarrow \frac{\hbar k_0}{m}$

So, as you can see this integral now represents wavepacket moving with group velocity (v_g)

phase velocity

$v_p = \frac{\omega(k)}{k}$

$v_p = \frac{\hbar k^2}{2mk} = \frac{\hbar k}{2m}$

$k = k_0$

$v_p = \frac{\hbar k_0}{2m}$

ok so, what we did,

dispersion relation : $\omega = \frac{\hbar k^2}{2m}$

phase velocity : $v_p = \frac{\hbar k}{2m}$

group velocity : $v_g = \frac{\hbar k}{m}$

hence, $v_g = \frac{v_p}{2}$

now, recall all of it ;

$v_p = \frac{\omega}{k}$

$v_g = \frac{d\omega}{dk}$

$k = \frac{2\pi}{\lambda} \longrightarrow \lambda = \frac{2\pi}{k}$

now,

$\omega = v_p k$

put this into v_g's equation;

$v_g = \frac{d}{dk}(v_p k)$

$\Rightarrow v_g = v_p + k\frac{dv_p}{dk}$

↳ in terms of λ :

using chain rule;

$v_g = v_p + k\frac{dv_p}{d\lambda}\frac{d\lambda}{dk}$

$\left\{ \lambda = \frac{2\pi}{k} \Rightarrow \frac{d\lambda}{dk} = \frac{d}{dk}(2\pi k^{-1}) \Rightarrow \frac{-2\pi}{k^2} \right\}$

$\frac{dv_g}{dk} = \frac{dv_p}{d\lambda}\left(\frac{-2\pi}{k^2}\right)$

↳ substitute ;

$v_g = v_p - \frac{2\pi}{k}\frac{dv_p}{d\lambda}$

$v_g = v_p - \lambda\frac{dv_p}{d\lambda}$

This tells us that the group velocity is generally less than the phase velocity in dispersive media, where $\frac{dv_p}{d\lambda}$ depends on λ. If v_p increases with λ, the group velocity decreases accordingly.

Hey again !!

remember double slit experiment ??

let's visit again

observations on screen :

⟶ when S_2 open

⟶ when S_1 open

⟶ when both slits open

- when we open both the slits, we see alternate bright and dark fringes on the screen
- when one the slits is closed, then this pattern doesn't appear rather we observe maxima in front of the slit that is open

let :

$\psi_1 = A\sin(\phi)$

$\psi_2 = A\sin(\phi+\delta)$ ← it's the phase difference

resultant wave would be,

$\psi = \psi_1 + \psi_2 = A\sin(\phi) + A\sin(\phi+\delta)$

$= A[\sin(\phi) + \sin(\phi+\delta)]$

for, $\delta = 2n\pi \quad (n \in \mathbb{Z})$

$\hookrightarrow \sin(\phi+\delta) = \sin(\phi)$

therefore,

$\psi = A[\sin\phi + \sin\phi] = 2A\sin(\phi)$

for, $\delta = (2n+1)\pi \quad (n \in \mathbb{Z})$

$\psi = A[\sin\phi - \sin\phi] = 0$

from this you can have a clear picture on this fringe pattern formation works !

Blackbody ??

↳ perfect absorber
↳ perfect emitter

and the model that any radiation comes out of this hole (B) is known as Blackbody radiation

and you know - I know, perfect black bodies don't exists in nature, we just can get closer to perfect !

let's take a metal piece into a dark room, and apply heat to it ...

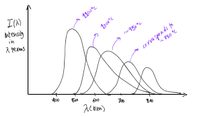
observation :-

Temp.	It's color
≈ 950°C	faint red
≈ 1150°C	dark orange
1200°C	orange
1400°C	yellow

} this gets fast till turned white in color and so on

same thing you observe in bulb

let's plot this



so, what we got ??

$\lambda_m \propto \frac{1}{T}$ ← Temperature

↓ wavelength, at which we get the peak intensity

$\Rightarrow \lambda_m = \frac{c}{T}$ ⟶ constant

$\Rightarrow \lambda_m = \frac{2.898 \times 10^{-3}}{T}\ m$ } wien's displacement law !!

another laws which was observed at this time : stephan's law ⟶

it states that ⟶

U (per unit time) (per unit area) $\propto T^4$

↓ ↓
Energy/energy emitted from a black body ; temperature

$\Rightarrow \frac{U}{At} \propto T^4 \quad \left\{ \sigma = 5.6703 \times 10^{-8} \frac{watt}{m^2 k^4} \right\}$

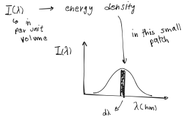
$\Rightarrow \frac{U}{At} = \sigma T^4$

now, remember that $I(\lambda)$ vs $\lambda(nm)$'s graph ?

to be continued ⟶

We'll see at the significance of $I(\lambda)$

$I(\lambda)$ ⟶ energy density



the first attempt to find the relation b/w $I(\lambda)$ & λ was done by Wien himself

he gave a distribution law

$\Rightarrow$ $I(\nu,T) = A\nu^3 e^{-B\nu/T}$ → wait, ν ?? don't we use λ ??

↓ spectral radiance in frequency & Temp terms

don't cry... "I'll include everything"

second attempt was made by Rayleigh & Jeans their way of defining was very sweet!

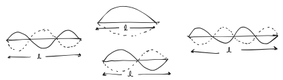
↳ In the Rayleigh-Jeans law for blackbody radiation, the radiation that filled the space is described as consisting of standing waves, which are produced by harmonic oscillators.

Wikipedia

Let's do a fun part!

remember this ? from high school ? I know you do 😉

standing waves can also be :



here, you must did notice, that at the ends of the system performing oscillations, A (Amplitude is 0 always)

which implies that $\frac{\lambda_n}{2} = \frac{L}{n}$; $n = 1, 2, 3, \ldots$

btw just to let you know Rayleigh-Jeans law :

$\frac{8\pi\nu^2}{c^3} KT = I(\nu,T)$

↓ we're looking at it's significance !!

why ?!

↳ because, $\frac{8\pi\nu^2}{c^3}$ is the no. of modes present in the ν ⟶ $(\nu+d\nu)$ frequency

Wikipedia

Also see this for :

https://en.m.wikipedia.org/wiki/Wien_approximation

https://www.feynmanlectures.caltech.edu/I_41_Ref.html

Thermal Physics by Charles Kittel & Herbert H. Kroemer

Introduction to the Modern Theory of Dielectrics by J. Van Vleck

Radiative Processes in Astrophysics by Rybicki & Lightman

Handout of Quantum - Introduction to Quantum Mechanics - pdf by Griffiths

Wikipedia

& do the same !!

consider a 3-D cavity (a box !!)

for a wave to form a standing wave, it wavelength must fit the box ...

allowed wavelengths :-

$\lambda_x = \frac{2L}{n_x}$, $\lambda_y = \frac{2L}{n_y}$, $\lambda_z = \frac{2L}{n_z}$

where $n_x, n_y, n_z \in \mathbb{N}$

each set of (n_x, n_y, n_z) defines a mode of oscillations

instead of working in λ terms let's move to wavevector $(\vec{k})$,

$\vec{k} = (k_x, k_y, k_z) = \left(\frac{\pi n_x}{L}, \frac{\pi n_y}{L}, \frac{\pi n_z}{L}\right)$

magnitude is :

$k = |\vec{k}| = \frac{\pi}{L}\sqrt{n_x^2 + n_y^2 + n_z^2}$

each mode corresponds to a point in 3D k-space (space of wavevectors)

- to be continued

∴ frequency and wavevector is related by ;

$$\nu = \frac{c}{\lambda} = \frac{ck}{2\pi} \quad \Rightarrow \quad \boxed{k = \frac{2\pi\nu}{c}}$$

let's count the no. of modes inside a sphere of radius k in k-space, each mode occupies a little cube in k-space of size $\left(\frac{\pi}{L}\right)^3$

so,

total no. of modes in magnitude $\leq k$;

$$N(k) = \frac{\text{Volume of 1/8}^{\text{th}}\text{ sphere of radius } (k)}{\text{Volume per mode}}$$

$$= \frac{1}{8} \times \frac{4}{3}\pi k^3 \div \left(\frac{\pi}{L}\right)^3$$

$$= \frac{L^3}{6\pi^2} k^3$$

The factor 1/8 is because only one positive octant is required.

modes per unit volume ;

$$\frac{N(k)}{V} = \frac{k^3}{6\pi^2}$$

now, differentiate to get modes per unit (k) ~ ~per unit volume.

$$\Rightarrow \quad \frac{dN}{dk} = \frac{d}{dk}\left(\frac{k^3}{6\pi^2}\right) = \frac{1}{6\pi^2} \times 3k^2 = \boxed{\frac{k^2}{2\pi^2}}$$

now, convert from k to ν ;

$$k = \frac{2\pi\nu}{c} \longrightarrow dk = \frac{2\pi}{c} d\nu$$

$$\Rightarrow \quad \frac{dN}{d\nu} = \frac{dN}{dk} \cdot \frac{dk}{d\nu} = \frac{k^2}{2\pi^2} \cdot \frac{2\pi}{c} = \boxed{\frac{k^2}{\pi c}}$$

now ; substitute k's value $\left(\frac{2\pi\nu}{c}\right)$ in it ;

$$\Rightarrow \quad \frac{dN}{d\nu} = \frac{1}{\pi c} \times \left(\frac{2\pi\nu}{c}\right)^2 = \boxed{\frac{4\pi\nu^2}{c^3}}$$

but remember each mode has 2 polarisations (eg. x & y dirn in E-field), therefore, multiply by 2 ;

$$\boxed{g(\nu) = \frac{8\pi\nu^2}{c^3}}$$ } no. of modes per unit frequency per unit volume

↓ ↳ see !! how attractive was it

this is how the N was calculated in Rayleigh-Jeans law

$$\{ g(\nu) = N \}$$

I am sorry, if I didn't mention above that the wien's distribution law doesn't work with low frequencies !

now, the problem with Rayleigh - Jeans law :-

$$I(\nu,T) = N \cdot E$$

$$I(\nu,T) = \frac{8\pi\nu^2}{c^3} \times kT$$

↳ diverges at higher frequencies !!

↳ which means that the energy emitted by a black body is ∞ !!

↳ this is also known as the " UV catastrophe "

well planck did a great job explaining the phenomenon finally !!

so, again according to Rayleigh & Jeans

$N = \frac{8\pi\nu^2}{c^3}$ ← modes per unit frequency per unit volume

& $\langle E \rangle = kT$ ← suitable thermodynamics

$$I(\nu,T) = \frac{8\pi\nu^2}{c^3} \times kT$$

as was basic standing waves can only have energy

$$\boxed{E_n = nh\nu}$$

let's derive planks radiation law mathematically ;

ok !!

How do we calculate an event's outcoming probability ??

..... BASIC PROBABILITY .....

$$P(E) = \frac{\text{desired outcome}}{\text{total no. of outcomes}}$$

$$\left(\text{eg. } P(\text{Head})_{\text{toss}} = \frac{1}{2}\right)$$

right ! _ cool !!

now, suppose i want to calculate the probability that a particular mode of vibrations have the energy (E_n)

it will come out as ;

$$P(n) = \frac{e^{-\beta n h\nu}}{\sum_{n=0}^{\infty} e^{-\beta n h\nu}}$$

(higher the energy of a standing wave, lesser the probability of its occurance)

→ to be continued

↳ as you can see,

$P(n)$ was nothing the normalization process that we simply do while calculating probability for an occurring event.

— the same we did ! —

and the avg. energy for the same ;

$$\langle E \rangle = \sum_{n=0}^{\infty} E_n P(n)$$

$$\Rightarrow \langle E \rangle = \frac{\sum_{n=0}^{\infty} E_n \left(e^{-E_n/kT}\right)}{\sum_{n=0}^{\infty} e^{-E_n/kT}}$$

$$(\text{as } E_n = nh\nu)$$

$$\Rightarrow \langle E \rangle = \frac{\sum_{n=0}^{\infty} nh\nu\, e^{-nh\nu/kT}}{\sum_{n=0}^{\infty} e^{-nh\nu/kT}}$$

to simplify, let's

do → $\left[e^{-h\nu/kT} = x \right]$

therefore ;

$$\langle E \rangle = \frac{\sum_{n=0}^{\infty} nh\nu\, x^n}{\sum_{n=0}^{\infty} x^n}$$

$$\Rightarrow \langle E \rangle = h\nu \frac{\sum_{n=0}^{\infty} n x^n}{\sum_{n=0}^{\infty} x^n}$$

expanding the term ;

$$\Rightarrow \langle E \rangle = \frac{h\nu \left[x + 2x^2 + 3x^3 + \cdots \right]}{\left[1 + x + x^2 + x^3 + \cdots \right]}$$

$$\Rightarrow \langle E \rangle = \frac{h\nu x \left[1 + 2x + 3x^2 + \cdots \right]}{\left[1 + x + x^2 + x^3 + \cdots \right]}$$

$$\left\{ 1 + 2x + 3x^2 + \cdots = \frac{1}{(1-x)^2} \right\}$$

$$\left\{ 1 + x + x^2 + x^3 + \cdots = \frac{1}{(1-x)} \right\}$$

therefore ,

$$\langle E \rangle = \frac{h\nu x \,(1-x)^2}{(1-x)^2} = \frac{h\nu x}{(1-x)}$$

divide (x) from both denominator and numerator

$$\langle E \rangle = \frac{\left(\frac{h\nu x}{x}\right)}{\left(\frac{1-x}{x}\right)} = \frac{h\nu}{\frac{1}{x} - 1} = \frac{h\nu}{x^{-1} - 1}$$

put x's substitution ,

$$\langle E \rangle = \frac{h\nu}{e^{h\nu/kT} - 1}$$

→ this is the value of avg. energy plank calculated using his discrete energy spectrum

↳ now to get the energy density of the black body radiation we need to do the product of avg. energy $\langle E \rangle$ with $N/g(\nu)$ modes of vibration !

which comes as ;

$$I(\nu,T) = N \cdot \langle E \rangle$$

$$= \frac{8\pi\nu^2}{c^3} \times \frac{h\nu}{e^{h\nu/kT} - 1}$$

$$I(\nu,T) = \frac{8\pi\nu^3}{c^3 \left[e^{h\nu/kT} - 1 \right]}$$

} Plank's radiation law

now as we've reached to it's end, just by applying a small amount of brain & gaming (CoD) drive from your precious inventory you'll be able derive the wien's & Rayleigh-Jean's law from plank's one !!

hint : frequency & binomial expansion !!

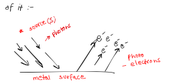
✢ photoelectric effect :

in this part there is honestly half stuff that you've about to learn for the first time elsewhere , high school has the half of it covered , just letting you know ...

→ to be continued

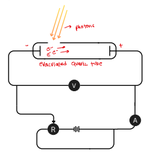
let's remind you of it :-

basically ,



↳ this is what happens , photons are emitted from the source (S) , gets hit to the metal surface - which is followed by the event of emission of electrons due to the energy that thing have granted ..

This is the actual setup of the experiment photoelectric effect :-



as you already know it's an instantaneous process and there are some observation regarding it

cutoff frequency / wavelength :-

every metal surface has a certain cutoff frequency through which the emission of electrons depends

eg. υ → frequency of the source
υ_0 → frequency of metal surface

if $(\upsilon \geq \upsilon_0)$ ⟶ photoelectric effect will happen

if $(\upsilon < \upsilon_0)$ ⟶ nope !! no photoelectric effect

{ and as $\lambda \propto \frac{1}{\upsilon}$ }

if $(\lambda \leq \lambda_0)$ ⟶ won with

if $(\lambda > \lambda_0)$ ⟶ no photoelectric effect

kinetic energy (K.E) depends on the wavelength (λ)

↳ $E = h\upsilon$

the minimum energy that is required to knock off the electron from a particular metal surface

↳ $\phi \Rightarrow \phi = h\upsilon_0$

$\{\upsilon_0 \rightarrow$ cutoff frequency $\}$

so ,

Kinetic energy (K.E)

⇒ $K.E = E - \phi = h\upsilon - h\upsilon_0$

⇒ $\boxed{K.E = h(\upsilon - \upsilon_0)}$

$(\upsilon > \upsilon_0$ ✓
$\upsilon < \upsilon_0$ ✗ $)$

• $E = h\upsilon$

$p = mv$

from special relativity , einstein derived an equation :-

$E^2 = p^2c^2 + m^2c^4$

for photon ,

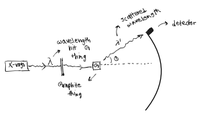
$E = pc$

from which the momentum would be ,

$p = \frac{E}{c}$

$\Rightarrow \boxed{p = \frac{h\upsilon}{c}}$

• compton effect :-

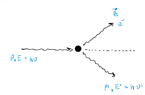


the main issue is that the wavelength which scattered after being hit to the graphite thing is not the same wavelength which was emitted from X-rays source !!

now this is a problem, as according to classical model, the scattered wavelength should've remained same ——— but it's not !!

well compton explained it by examining the system as it's having a collision event in it .

therefore :-



- to be continued

conservation of momentum :

before scattering $\{ \vec{p} + 0 = \vec{p_e} + \vec{p'} \}$ after scattering

$\rightsquigarrow \vec{p_e} = \vec{p} - \vec{p'}$

$\Rightarrow |\vec{p_e}|^2 = |\vec{p} - \vec{p'}|^2$

$\Rightarrow |\vec{p_e}|^2 = \vec{p}^2 + \vec{p'}^2 - 2\vec{p}\vec{p'}\cos\phi$

$= \left(\frac{h\nu}{c}\right)^2 + \left(\frac{h\nu'}{c}\right)^2 - 2\frac{h^2}{c^2}(\nu\nu'\cos\phi)$

$\Rightarrow |\vec{p_e}|^2 = \frac{h^2}{c^2}(\nu^2 + \nu'^2 - 2\nu\nu'\cos\phi)$ → (i)

energy :

$E + mc^2 = E' + E_e'$

$\Rightarrow h\nu + mc^2 = h\nu' + E_e'$

$\{(Ec)^2 = p^2c^2 + m^2c^4\}$ → substitute

$\Rightarrow h\nu + mc^2 = h\nu' + \sqrt{p_e^2c^2 + m^2c^4}$

from equation (i) above substitute $(pc)^2$

$\Rightarrow h\nu + mc^2 = h\nu' + \sqrt{h^2(\nu^2 + \nu'^2 - 2\nu\nu'\cos\phi) + m^2c^4}$

$\Rightarrow h\nu + mc^2 = h\nu' + h\sqrt{(\nu^2 + \nu'^2 - 2\nu\nu'\cos\phi) + \frac{m^2c^4}{h^2}}$

$\Rightarrow \nu + \frac{mc^2}{h} = \nu' + \sqrt{\nu^2 + \nu'^2 - 2\nu\nu'\cos\phi + \frac{m^2c^4}{h^2}}$

$\Rightarrow (\nu - \nu') + \frac{mc^2}{h} = \sqrt{\nu^2 + \nu'^2 - 2\nu\nu'\cos\phi + \frac{m^2c^4}{h^2}}$

→ take the square and simplify the above stuff and you'll get :

$$\frac{1}{\nu'} - \frac{1}{\nu} = \frac{h}{mc^2}(1 - \cos\phi)$$

which is ...

$$\frac{1}{\nu'} - \frac{1}{\nu} = \frac{h}{mc^2}(1 - \cos\phi)$$

$\{c = \nu\lambda\}$

therefore :

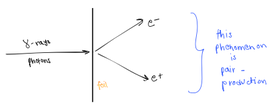
$$\frac{1}{\nu'} - \frac{1}{\nu} = \frac{h}{mc^2}(1 - \cos\phi) = \lambda' - \lambda$$

$\{(\lambda' - \lambda)_{max} = \frac{2h}{mc} = 0.05\ \text{Å}\}$ for visible light

$\Rightarrow 5 \times 10^{-12}$ m

if the energy of photon decreases, it is known as crompton effect but if it increases, it's the inverse crompton effect .... in active galaxies, inverse crompton effect takes place.

→ pair production :



this phenomenon is pair - production

pair - production is basically the generation of matter - antimatter pairs from neutron bosons and to perform this process, tail is required as it provides coloumb fields for the relevance.

eg) electron ⟷ positrons
muons ⟷ anti-muons
proton ⟷ anti-proton

from the diagram mentioned above,

↳ $\gamma \longrightarrow e^- + e^+$ } what was shown

so, as we know, charge must be conserved ...
also the lepton numbers, an important quantum number to pay attention to ...

↓

well, what are leptons ??

↳ ...

~ to be continued

↳ leptons are the elementary particles with half integer spin which do not undergo strong interactions

accord to standard model, there are six leptons

↳ $e^-, \mu^-, \tau^-, \nu_e, \nu_\mu, \nu_\tau$

lepton nos associated with them :-

lepton no = +1	lepton no = -1
electron	positron
muon	anti muon
Tau	anti Tau
electron nutrino	positron nutrino
muon nutrino	anti muon nutrino
Tau nutrino	anti Tau nutrino

all other particles, including bosons has lepton no as = 0

- lepton no must be conserved for pair production

lets calculate the minimum energy for the process of pair production to initiate :-

$\Rightarrow E_{min} = 2mc^2$

$\Rightarrow E_{min} = 2 \times 0.511\ MeV$

$\Rightarrow E_{min} = 1.022\ MeV$ } minimum energy for pair formation of $(e^- + e^+)$

hence, any photon which acquires lesser energy than 1.022 MeV, cannot perform respective pair - production.

now, if you put E value in planck's formulation

$E = h\nu$

$\Rightarrow \nu = 2.47 \times 10^{20}\ Hz$

and from $(c = \lambda\nu)$

$\Rightarrow \lambda = 1.2 \times 10^{-12}\ m$

now the debate on particle's nature has a wave part too ...

de broglie waves (you know all this) trust me !!

$\lambda = \frac{h}{p}$ → you know it ...

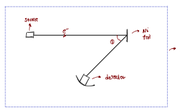
$E = h\nu$ (planck's law) $\quad E = pc$ (Einstein's one)

$h\nu = pc$

$\Rightarrow \frac{hc}{\lambda} = pc$

$\Rightarrow \lambda = \frac{h}{p}$ ✓ you know it (nothing's new)

next, 2 scientists, Davisson & Germer were performing an experiment to look out for studying the angular distribution of electrons when scattered through a nickel foil



→ Copper K α characteristic

this experiment was supposed to be closed but as air got ionized, due to which Ni foil got oxidized

therefore, they treated it with heat and resumed with the experiment

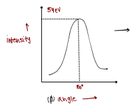
and what they observed wasn't expected

they saw, almost all the angles - there was only one angle at which most of the electrons were being scattered !

and that angle was a function of initial kinetic energy of the electrons

which apparently evolved as :-

→ to be continued



experiment recorded in data like this

and if you notice it'll remind you of X-ray diffraction's graph

for further details, you can refer to →

https://en.wikipedia.org/wiki/Davisson%E2%80%93Germer_experiment

hence, davisson - germer proved de-broglie hypothesis

∴ smaller the particles, more prominent the wave nature becomes

→ further shift to get from the usual formulations :

as, $p = \sqrt{2mE}$

accordingly you can write

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}}$$

also as, $pc = \sqrt{E^2 - m^2c^4}$ by some previous steps we performed

therefore in usage respect :

$$\lambda = \frac{hc}{\sqrt{E^2 - m^2c^4}}$$

don't loose on basic terms :

like

$KE = \frac{p^2}{2m}$, $p = \sqrt{2mE}$; precisely $\left\{\sqrt{2m(KE)}\right\}$

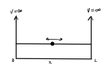
$\lambda = \frac{2\pi}{k}$ and such whatever so on ...

you got this !!

also, $h\nu = \gamma mc^2$

where $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$ relativistic

∴ particle in a box



a particle in a 1D infinite potential well

inside the box potential (V) is 0 (across L)
or outside the box ($V = \infty$; $x > L$, $x < 0$)

we assume that the walls have an potential energy to ensure there is 0 probability for particle to passing at the walls

the time independent Schrodinger's equation,

$$\frac{-\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$$

now, as V is 0 everywhere inside ;

↳ $\frac{-\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x)$

→ $\psi(x) = A\sin(kx) + B\cos(kx)$

↳ you well remember this

let's apply some boundary conditions as at $x=0$ & $x=L$ probability of particle is 0

when $x = 0$ $\sin(0) = 0$
$\cos(0) = 1$

therefore B must be 0

to satisfy the condition.

giving :

$\psi(x) = A\sin(kx)$

we can now solve for our constants (A & k)

∴ solving for k →

$$\frac{d\psi}{dx} = kA\cos(kx)$$

$$\frac{d^2\psi}{dx^2} = -k^2A\sin(kx)$$

as $\psi(x) = A\sin(kx)$;

$$\frac{d^2\psi}{dx^2} = -k^2\psi$$

if we solve for k comparing with the schrodinger's equation, we'll get ;

$$k = \frac{(8\pi^2 mE)^{1/2}}{h}$$

plug k in $\psi(x)$;

$$\psi = A\sin\left[\left(\frac{8\pi^2 mE}{h^2}\right)^{1/2} x\right]$$

solving for A ;

∴ to be continued

(when $x=L$) →

$$0 = A\sin\left[\left(\frac{8\pi^2 mE}{h^2}\right)^{\frac{1}{2}} L\right]$$

this is only true if;

$$\left(\frac{8\pi^2 mE}{h^2}\right)^{\frac{1}{2}} L = n\pi$$

where $n = 1, 2, 3 \ldots$

plugging this back,

$$\psi = A\sin\frac{n\pi}{L}x$$

combining we know that

probability of finding the particle inside the box $= 1$, which means outside the box probability is 0. When we find the probability and set it equals to 1, we're normalising the wavefunction !

$$\int_0^L \psi^2 \cdot dx = 1$$

for our system normalisation looks like →

$$A^2 \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx = 1$$

using the solution of this integral from integral table

we find our normalisation constant A;

$$A = \sqrt{\frac{2}{L}}$$

which is

$$\psi = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$$

solving for E results in the allowed energies for a particle in a box :

$$E_n = \frac{n^2 h^2}{8mL^2}$$

this important result tells us ;

→ Energy is quantized for a particle

→ lowest possible energy of a particle is NOT 0

hence called *zero-point-energy* and means particle can never be at rest because it always has some kinetic energy.

some straight forward stuff you need to look out for

$$\langle \hat{B} \rangle = \frac{\int \psi(x) \cdot \hat{f}(x) \cdot dx}{\int \psi(x) \cdot dx}$$

$$\langle x \rangle = \frac{\int x \cdot f(x) \cdot dx}{\int f(x) \cdot dx}$$

$$\langle x^2 \rangle = \frac{\int x^2 \cdot f(x) \cdot dx}{\int f(x) \cdot dx}$$

- expectation values:

the average position ($\bar{x}$) of a no. of identical particles distributed along x-axis, such a way that there are N_1 particles at x_1, N_2 particles at x_2 and so on..

the average position in this case is the same as the centre of mass of the distribution, and so

$$\bar{x} = \frac{N_1 x_1 + N_2 x_2 + N_3 x_3 + \cdots + N_i x_i}{N_1 + N_2 + N_3 + \cdots + N_i}$$

$$\Rightarrow \bar{x} = \frac{\sum N_i x_i}{\sum N_i}$$

when we are dealing with single particle, we must replace the no. N_i of particles at x_i by the probability P_i that the particle be found in an interval dx at x_i, the probability is:-

$$P_i = |\psi_i|^2 \cdot dx$$

where ψ_i is the particle wave function evaluated at $x = x_i$. Making this substitution and changing the summations to integrals, we see that the expectation value of the position of the single particle is →

$$\langle x \rangle = \frac{\int_{-\infty}^{\infty} x|\psi|^2 dx}{\int_{-\infty}^{\infty} x|\psi|^2 \cdot dx}$$

and if ψ is normalised wave function

~ to be continued

then the value of denominator of the expectation value will be 1

hence, in such case,

$$\langle x \rangle = \int_{-\infty}^{\infty} x|\Psi|^2 \, dx$$

similarly was figured for $\langle f(x(t)) \rangle$ for some quantities (eg: potential energy $(U(x))$)

operators :-

another way to find expectation values,

A hint as to the proper way to evaluate <p> and <E> comes from differentiating the free particle wave function with respect to x and to t.

Hobart

we find that :

$$\frac{\partial \Psi}{\partial x} = \frac{i}{\hbar} p\Psi$$

$$\frac{\partial \Psi}{\partial x} = -\frac{i}{\hbar} E\Psi$$

↳ can be written in preferred forms ;

$$p\Psi = \frac{\hbar}{i} \frac{\partial}{\partial x} \Psi$$

$$E\Psi = i\hbar \frac{\partial}{\partial t} \Psi$$

now, evidently p in some sense corresponds to the differential operator $\rightarrow \left(\frac{i}{\hbar} \frac{\partial}{\partial x}\right)$

similarly, $E \rightarrow \left(i\hbar \frac{\partial}{\partial t}\right)$

it is customary to denote operators by using a caret, so that $\hat{p}$ is the operator that corresponds to momentum p and $\hat{E}$ is the operator corresponds to total energy E

the operators are

$$\text{momentum} \longrightarrow \hat{p} = \frac{\hbar}{i} \frac{\partial}{\partial x}$$

$$\text{total-energy} \longrightarrow \hat{E} = i\hbar \frac{\partial}{\partial t}$$

though we have only shown that the correspondences expressed in Equations stated above, hold for free particles they're entirely general results who's validity is same as that of Schrodingar's equation

to support this we can replace :-

$$E = KE + U$$

with the operator equation

$$\hat{E} = \hat{KE} + \hat{U}$$

here, $\hat{U}$ is just $U(\Psi)$ and KE is expressed in terms of momentum - (p) by

$$KE = \frac{p^2}{2m}$$

kinetic-energy operator :-

$$\hat{KE} = \frac{\hat{p}^2}{2m} = \frac{1}{2m}\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$$

now $\hat{E} = \hat{KE} + \hat{U}$ looks :-

$$i\hbar \frac{\partial}{\partial t} = -\frac{\hbar^2}{2m} \cdot \frac{\partial^2}{\partial x^2} + U$$

now, when we multiply $\Psi = \Psi$ identity :-

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \cdot \frac{\partial^2 \Psi}{\partial x^2} + U\Psi$$

which is Schrodingar's equation !

for further info I insist to please have a look at references

Concepts of Modern Physics by Arthur Beiser

Hobart

Griffiths

• Schrodingar's steady state form equation :-

now i've stated, how to get Schrodingar's equation at many instances, a brief again,

$$\Psi = Ae^{-(i/\hbar)(Et - px)} = Ae^{-(i/\hbar)E_n t} e^{(i/\hbar)px}$$

$$= \psi_n e^{(i/\hbar)px}$$

evidently Ψ is the product of a time-dependent function $e^{-(i/\hbar)Et}$ and a position dependent Ψ. as it happens, the time variations of all wave functions of particles acted on by forces independent of time have the same form as that of an unconfined particle.

↓ to be continued

substituting the Ψ of eq $\{\Psi = \psi e^{-(iE/\hbar)t}\}$
into the time dependent form
of schrödinger's equation;

$$E\psi e^{-(iE/\hbar)t} = -\frac{\hbar^2}{2m} e^{-(iE/\hbar)t} \frac{\partial^2 \psi}{\partial x^2} + U\psi e^{-(iE/\hbar)t}$$

dividing through by the common exponential factor
gives:

Steady state eq:
one-dimension
S.E.

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{2m}{\hbar^2}(E-U)\psi = 0$$

Steady state eq:
three-dimension
S.E.

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} + \frac{2m}{\hbar^2}(E-U)\psi = 0$$

for further please check references ...

Eigenvalues & Eigenfunctions:

the values of E_n for which schrödinger steady-state
equation can be solved are known as
eigenvalues.

and the corresponding ware functions ψ_n are known
as eigenfunctions.

the discrete energy levels of the hydrogen atom

$$E_n = \frac{-me^4}{32\pi^2\epsilon_0^2\hbar^2}\left(\frac{1}{n^2}\right) \quad n = 1, 2, 3, \cdots$$

are an example of a set of eigenvalues.

other than total energy, dynamical variable which is
quantized, is angular momentum (L)
in the hydrogen atom's case, we shall find that
the eigenvalues of the magnitude of the total angular
momentum are specified by

$$L = \sqrt{l(l+1)}\hbar \qquad l = 0, 1, 2, 3 \cdots (n-1)$$

ofcourse, dynamical variable G may not be quantized

Eigenvalue equation;

$$\hat{G}\psi_n = G_n\psi_n$$

in view of $\hat{E} = \hat{K}E + \hat{U}$, total energy operator
can also be written as

$$\hat{H} = -\frac{\hbar^2}{2m}\cdot\frac{\partial^2}{\partial x^2} + \hat{U}$$

and is called the hamiltonian operator

because it is reminiscent of the hamiltonian function
in Advanced classical mechanics, which is
an expression for total energy of the system
in terms of coordinates and momenta only;
due to which the steady-state schrödinger equation
can also be stated as:

$$\hat{H}\psi_n = E_n\psi_n$$

mentioning important inclusives ;

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$$

$$\Delta E \cdot \Delta t \geq \frac{\hbar}{2}$$

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

• Atomic structure :-

Rutherford scattering formula

↳ $N(\theta) = \frac{Nint z^2 e^4}{(8\pi\epsilon_0)^2 r^2 KE^2 \sin^4(\theta/2)}$

no. of alpha particles per unit area, that reached the screen at a scattering angle θ

$N_t \longrightarrow$ total no. of alpha particles which reach the screen

$n \longrightarrow$ no. of atom per unit volume in the foil

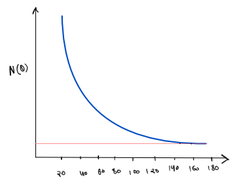
$Z \longrightarrow$ atomic no. of a foil atom

$r \longrightarrow$ distance of the screen from the foil

$KE \longrightarrow$ kinetic energy of the alpha particles

$t \longrightarrow$ foil thickness

the curve $N(\theta)$ follows :-



nuclear dimensions

An alpha particle will have its smallest R when it approaches a nucleus head on, which will be followed by a 180° scattering.

at the instance of closest approach the initial kinetic energy KE of the particle is entirely converted to electrical potential energy

and so at that instant

$$KE_{initial} = PE = \frac{1}{4\pi\epsilon_0} \frac{2Ze^2}{R}$$

since the charge of an alpha particle is $2e$ and that of the nucleus is Ze hence,

distance of closest approach

↳ $R = \frac{2Ze^2}{4\pi\epsilon_0 KE_{initial}}$

$f_c = \frac{mv^2}{r}$ (centripetal force)

$f_c = \frac{1}{4\pi\varepsilon_0}\left(\frac{q^2}{r^2}\right)$ (coulomb's force)

$f_c = f_e$

you'll get, $v = \frac{e}{\sqrt{4\pi\varepsilon_0 mr}}$

$KE = \frac{1}{2}mv^2$, $PE = -\frac{e^2}{4\pi\varepsilon_0 r}$

$E = KE + PE$

$E = \frac{1}{2}mv^2 - \frac{e^2}{4\pi\varepsilon_0 r}$

$E = \frac{1}{2}m\left(\frac{e}{\sqrt{4\pi\varepsilon_0 mr}}\right)^2 - \frac{e^2}{4\pi\varepsilon_0 r}$

$E = \frac{1}{2}\cdot\frac{e^2}{4\pi\varepsilon_0 r} - \frac{e^2}{4\pi\varepsilon_0 r}$

$E = \frac{e^2}{8\pi\varepsilon_0 r} - \frac{e^2}{4\pi\varepsilon_0 r}$

$E = -\frac{e^2}{8\pi\varepsilon_0 r}$

↳ total energy for hydrogen atom.

problem is, just put r from above expression...

$r = -\frac{e^2}{8\pi\varepsilon_0 E}$

Bohr, specified those allowed orbits he known already,

→ orbital electron quantization,

$\lambda = \frac{h}{mv}$

substitute v ;

$\lambda = \frac{h}{m\left(\frac{e}{\sqrt{4\pi\varepsilon_0 mr}}\right)}$

$\lambda = \frac{h\sqrt{4\pi\varepsilon_0 mr}}{me}$

$\lambda = \frac{h}{e}\sqrt{\frac{4\pi\varepsilon_0 r}{m}}$

• condition for orbital stability ;

$2\pi r = n\lambda$

$\therefore 2\pi r = n\left(\frac{h}{e}\sqrt{\frac{4\pi\varepsilon_0 r}{m}}\right)$

$v_n = n\frac{h}{e}\sqrt{\frac{4\pi\varepsilon_0 r_n}{m}}$

$\frac{v_n}{\sqrt{r_n}} = \frac{nh}{e}\sqrt{\frac{4\pi\varepsilon_0}{m}}$

$v_n = \frac{n^2h^2}{e^2}\frac{4\pi\varepsilon_0}{m}\frac{mv_n^2}{2}$

→ $v_n = \frac{nh}{e^2\pi m}\frac{4\pi\varepsilon_0 mv_n^2}{2}$ ←

Bohr realise ,

$a_0 = v_n = e \cdot E \cdot 9 \cdot \hat{n}$

where value are previously obtained,

$v_n = \frac{2\pi e^2}{nh}$

• energy levels & spectra.

$E = \frac{-e^2}{8\pi\varepsilon_0 r_n}$

substitute r_n from above expression,

$E = \frac{-e^2}{8\pi\varepsilon_0\left(\frac{n^2h^2\varepsilon_0}{\pi me^2}\right)}$

$E_n = \frac{-e^2\cdot\pi me^2}{8\pi\varepsilon_0 n^2h^2\varepsilon_0}$

$E_n = -\frac{me^4}{8n^2\varepsilon_0^2h^2}$

$E_n = -\frac{me^4}{8n^2\varepsilon_0^2h^2}\left(\frac{1}{n^2}\right)$

$E_n = E_1\left(\frac{1}{n^2}\right)$

after this energy part comes for spectral line → science is be known in gen !!

• correspondence principal :

→ system will be the quantum numbers, and classical physics approaches classical physics.

f_c : $v = \frac{e}{\sqrt{4\pi\varepsilon_0 mr}}$

frequency = $\frac{\text{electron speed}}{\text{orbit circumference}} = \frac{v}{2\pi r}$

$= \frac{e}{\sqrt{4\pi\varepsilon_0 mr}\,(2\pi r)}$

$f = \frac{e}{8\pi\sqrt{\pi\varepsilon_0 mr^3}}$

• atom's is also written for frequency of photon among particle condition

→ (it even worth paying attention ! full paper by William-Planck from Bohr's book)

• under nuclear motion, it has expression for "reduced mass"

↳ $m' = \frac{mM}{m+M}$

→ energy levels corrected for nuclear motion ;

$E_n = -\frac{me^4}{8\varepsilon_0^2h^2}\left(\frac{1}{n^2}\right) = \frac{m}{m}\left(\frac{E_n}{n^2}\right)$

you'll be able to understand everything from the notes if you'll compare this from my notes ! as everything except is in the matter wave acceleration !!

now, I want you to understand how Schrödinger equation came formed in 1 dimension before jumping towards multi-dimension quantum field theory...

recalling that basic wave equation ;

$\frac{\partial^2\psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$

just in case if you forget it's significance,

to be continued →

So, if you remember we used to work with a very beautiful representation of general

$\psi(x,t) = f(x-ct) + h(x+ct)$

↳ a very general representation of travelling waves which basically consist of case to which we can refer to on the basis of our need

{ first assignment had the solution for its standing wave case }

so, we're about to do the same kind of a thing

So, let

$\psi = A f(x-vt)$

now, first we take t as constant and diff. it w.r.t x

$\frac{\partial \psi}{\partial x} = A \frac{\partial f(x-vt)}{\partial (x-vt)} \cdot \frac{\partial (x-vt)}{\partial x}$

Can write it as,

$\longrightarrow \frac{\partial \psi}{\partial x} = A f'$

again diff wrt x,

$\frac{\partial^2 \psi}{\partial x^2} = A \frac{\partial f'}{\partial (x-vt)} \cdot \frac{\partial (x-vt)}{\partial x}$

Can write it as,

$\longrightarrow \boxed{\frac{\partial^2 \psi}{\partial x^2} = A f''}$

now, diff ψ wrt to t with having x as constant

$\frac{\partial \psi}{\partial t} = A \frac{\partial f(x-vt)}{\partial (x-vt)} \cdot \frac{\partial (x-vt)}{\partial t}$

$\frac{\partial \psi}{\partial t} = A f' \frac{\partial (x-vt)}{\partial t}$

$\frac{\partial \psi}{\partial t} = A f'(-v)$

again diff. wrt t,

$\frac{\partial^2 \psi}{\partial t^2} = -Avf''(-v)$

$\boxed{\frac{\partial^2 \psi}{\partial t^2} = +v^2 A f''}$

now, from above equation,

substitute Af'' equality;

$\boxed{\frac{\partial^2 \psi}{\partial t^2} = v^2 \frac{\partial^2 \psi}{\partial x^2}}$

now, for free particle, {no interaction}

$\psi = Ae^{i(kx-\omega t)}$

$E = h\nu = \frac{1}{2}mv^2 = \frac{p^2}{2m}$

$E = \frac{p^2}{2m} = \hbar\omega$

$\cdot\ p = \hbar k$

now, put these values into the equation,

$\psi = Ae^{i(kx-\omega t)}$

$\hookrightarrow \psi = Ae^{\frac{i}{\hbar}\left(\frac{p}{\hbar}x - \frac{E}{\hbar}t\right)}$

$\hookrightarrow \psi = Ae^{\frac{i}{\hbar}(px-Et)}$

diff. it w.r.t (x),

$\frac{\partial \psi}{\partial x} = Ae^{\frac{i}{\hbar}(px-Et)} \cdot \left(\frac{i}{\hbar}p\right)$

$\frac{\partial \psi}{\partial x} = \psi\left(\frac{i}{\hbar}p\right)$

$\boxed{-i\hbar \frac{\partial \psi}{\partial x} = p\psi}$

now diff. again w.r.t (x),

$\frac{\partial^2 \psi}{\partial x^2} = A\left(\frac{i}{\hbar}p\right)\left(\frac{i}{\hbar}p\right) e^{\frac{i}{\hbar}(px-Et)}$

$\frac{\partial^2 \psi}{\partial x^2} = -\frac{p^2}{\hbar^2} Ae^{\frac{i}{\hbar}(px-Et)}$

$\boxed{\frac{\partial^2 \psi}{\partial x^2} = -\frac{p^2}{\hbar^2}\psi}$

now, diff it w.r.t (t),

$\frac{\partial \psi}{\partial t} = Ae^{\frac{i}{\hbar}(px-Et)} \cdot \left(-\frac{i}{\hbar}E\right)$

$\boxed{\frac{\partial \psi}{\partial t} = -\frac{i}{\hbar}E\psi}$

$E\psi = -\frac{\hbar}{i}\frac{\partial \psi}{\partial t} = i\hbar\frac{\partial \psi}{\partial t}$

for free particle $\longrightarrow$ $E = \frac{p^2}{2m}$

$\hookrightarrow E\psi = \frac{p^2}{2m}\psi$

now, substituting $p^2\psi$'s equality from previously derived terms,

$E\psi = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} = i\hbar\frac{\partial \psi}{\partial t}$

$E = \frac{p^2}{2m} + V(x)$

$\hookrightarrow E\psi = \frac{p^2}{2m}\psi + V(x)\psi$

hence,

$\boxed{-\frac{\hbar^2}{2m}\cdot\frac{\partial^2 \psi}{\partial x^2} + V(x)\psi = i\hbar\frac{\partial \psi}{\partial t}}$

↳ Schrödinger equation $\longrightarrow$

• Potential Barrier / Quantum Tunneling :-

now, the region below 1 shows $E < V_0$ case of for B
& the region above 1 shows $E > V_0$ case of for B

and when the only difference it replicate further being the particle in the box is that (PIB) used to have $V(x) = \infty$ at the two sides and $V(x) = 0$ in the middle !

$$\frac{-\hbar^2}{2m} \cdot \frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x) \quad \} \text{ TISE}$$

now, for specifically R-A & R-C (Region A & C)
as $V(x) = 0$
for those we've got SE as,

$$\frac{-\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} + 0 = E\psi(x) \quad \} \text{ for R-A \& R-C}$$

now to get partial Diff. equations ;

$$\rightarrow \frac{d^2\psi(x)}{dx^2} = E\psi(x) \cdot \frac{2m}{\hbar^2}$$

$$\rightarrow \frac{d^2\psi(x)}{dx^2} = -\frac{2mE}{\hbar^2}\psi(x)$$

now, we define (k) ;

$$\frac{d^2\psi(x)}{dx^2} = -k^2\psi(x)$$

$$\{\therefore k^2 = \frac{2mE}{\hbar^2}\}$$

$$\{\therefore k = \sqrt{\frac{2mE}{\hbar}}\}$$

taking k to the other side,

$$\frac{d^2\psi(x)}{dx^2} + k^2\psi(x) = 0 \quad \} \text{ a differential equation, as was mostly studied in particle in the box problem}$$

now, why this equation possibly can have similar solution like particle in the box'le one,
→ due to boundary conditions differences !

→ despite $\psi(x)$ differ for regions ;

$$\psi_1(x) = Ae^{ikx} + Be^{-ikx} \quad \} \text{ for R-A}$$

$$\psi_2(x) = Ce^{ikx} + De^{-ikx} \quad \} \text{ for R-C}$$

→ both have same (k), due to similar potentials !

now, let's define the same for R-B (region-B)

$$\rightarrow -\frac{\hbar^2}{2m} \cdot \frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$$

$$\rightarrow \{\text{as } V(x) = V_0 \text{ in R-B}\}$$

$$\rightarrow -\frac{\hbar^2}{2m} \cdot \frac{d^2\psi(x)}{dx^2} + V_0\psi(x) = E\psi(x)$$

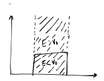
as we can see for regions of $V(x) = 0$
we used imaginary exponentials for $\psi(x)$ eqns
but now we need to use real exponentials
and to acknowledge it's significance ;

$f(x) = e^x + e^{-x}$ $\qquad$ $g(x) = e^{ix} + e^{-ix}$

sum of real exps
→ it excepts a narrow domain !

→ steps attempting
we'll use this a reference
the behavior at this
is a function defines from same matter !

sum of imaginary exp
→ as we already know behavior of $\psi(x)$ functions
→ it shows stationary waves
→ which is unaltered by potentials



now,
for $E < V_0$
as we can't consider
$V(x)$ from there

we do,

$$\frac{-\hbar^2}{2m} \cdot \frac{d^2\psi(x)}{dx^2} + V_0\psi(x) - E\psi(x) = 0$$

$$\rightarrow -\frac{\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} + (V_0 - E)\psi(x) = 0$$

→ to be continued >

so,
we had

$$\rightarrow \frac{-\hbar^2}{2m}\cdot\frac{d^2\psi(x)}{dx^2} + (V_0 - E)\psi(x) = 0$$

now, do

$$\frac{2m}{\hbar^2}\left[\frac{-\hbar^2}{2m}\cdot\frac{d^2\psi(x)}{dx^2} + (V_0 - E)\psi(x)\right] = (0)\cdot\frac{2m}{\hbar^2}$$

$\rightarrow$ now simply,

$$-\frac{d^2\psi(x)}{dx^2} + \underbrace{\left[(V_0 - E)\psi(x)\cdot\frac{2m}{\hbar^2}\right]}_{\text{from here we'll get Klein (K}_0\text{)}} = 0$$

$$\left\{\frac{2m(V_0 - E)}{\hbar^2} = K_0^2\right\}$$

$$\rightarrow \left\{K_0 = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}\right\}$$

hence for $E < V_0$ condition ;

$$\left(-\frac{d^2\psi(x)}{dx^2} + K_0^2\psi(x) = 0\right)$$

hence ;

$$\psi_{R,B}(x) = Fe^{K_0 x} + Ge^{-K_0 x} \quad \} \text{ for R-B}$$

for the case above the barrier ($E > V_0$)

$$\hookrightarrow -\frac{\hbar^2}{2m}\cdot\frac{d^2\psi(x)}{dx^2} + (E - V_0)\psi(x) = 0$$

$$\text{or } -\frac{d^2\psi(x)}{dx^2} + \left(\frac{2m(E - V_0)}{\hbar^2}\right)\psi(x) = 0$$

$$\left(-\frac{d^2\psi(x)}{dx^2} + K_0^2\psi(x) = 0\right) \quad \} \text{ for } (E > V_0) \text{ (Klein/Ka)}$$

$$\left\{K_0^2 = \frac{2m(E - V_0)}{\hbar^2}\right\}$$

$$\left\{K_0 = \sqrt{\frac{2m(E - V_0)}{\hbar^2}}\right\}$$

B/W $\longrightarrow$ as the region above B is out of the potential barriers we'll be using complex exponentials!!

hence,

$$\psi_{B,A} = Ae^{ikx} + \underbrace{Be^{-ikx}}_{\downarrow} \quad \} \text{ for (B-A) part!}$$

Let's split the terms for classism ;

(R-A)	(R-B)	(R-C)
$\psi_{RA} = Ae^{ikx} + Be^{-ikx}$ ↳ right part of it $\rightarrow Ae^{ikx}$ ↳ left part of it $\rightarrow Be^{-ikx}$	$\psi_{RB} = Fe^{kx} + Ge^{-kx}$ $\psi_{RB} = Fe^{kx} + Ge^{-kx}$ ψ_{RB} (right part) $\rightarrow Fe^{kx}$ ψ_{RB} (left part) $\rightarrow Ge^{-kx}$	$\psi_C(x) = Ce^{ikx} + De^{-ikx}$ ↳ right part of it $\rightarrow Ce^{ikx}$ ↳ left part of it $\rightarrow \underline{De^{-ikx}} = 0$ due to $R \downarrow$ $D = 0$
→ wave can possibly travel in both directions ⟸ it's possible when particle reflected back	→ wave can possibly travels in both directions ⟸ it's possible when particle penetrates but completely but prior to reflection	→ because D must be 0, as wave can't propagate in the left dir'n now, its common sense but still if you feel → why it can't travel in left dir'n ? → don't hesitate asking me out !

And well as what apparent conditions, at which we know a possibility comes to certainty ? :-

eg). $\psi_A(x) = Ae^{ikx} + Be^{-ikx}$

$$\left(\begin{array}{l} \text{→ right} \rightarrow Ae^{ikx} \\ \text{→ left} \rightarrow Be^{-ikx} \end{array}\right\} \text{A \& B represent relative probabilities in these dirs}$$

at $B = 0 \longrightarrow$ we'll be left with

$$\psi(x) = Ae^{ikx}$$

for this the probability density function would be $\Rightarrow P(x) = \psi\psi^* = Ae^{ikx}\cdot Ae^{-ikx}$

$$\rightarrow A^2 e^{(ikx - ikx)}$$

$$\rightarrow A^2 e^0$$

$$\rightarrow A^2 \quad \} \text{ which mathematically justifies that at } B = 0 \text{ wave can only travel right !}$$

now, we apply boundary conditions ;

$\rightarrow$ wavefunction & its derivative need to be continuous

$\rightarrow$ wavefunction must be normalizable

means

$$\int_{-\infty}^{\infty}\psi(x)\psi^*(x)\cdot dx = 1$$

$\longrightarrow$ to be continued ...

For the first condition;

$$\psi_A(x) = \psi_{B,b}(x)$$

$$Ae^{ik_0x} + Be^{-ik_0x} = Fe^{k_0x} + Ge^{-k_0x}$$

Since we're examining $(x=0)$

put 0 →

and we get :

$$A + B = F + H \longrightarrow (1)$$

Also, derivative (cont') :

$$\left.\frac{\partial \Psi_R}{\partial x}\right|_{x=0} = \left.\frac{\partial \Psi_{RR}}{\partial x}\right|_{x=0}$$

$$\frac{\partial}{\partial x}\left(Ae^{ikx} + Be^{-ikx}\right) = \frac{\partial}{\partial x}\left(Fe^{ikx} + He^{-ikx}\right)$$

$$\rightarrow ik\left(Ae^{ikx} - Be^{-ikx}\right) = ik\left(Fe^{ikx} - He^{-ikx}\right)$$

put $(x=0)$

$$ik_1(A - B) = ik_2(F - H) \longrightarrow (2)$$

now, for the end tip of the barrier to ensure continuity for Ψ_{RR} & Ψ_L

$$\Psi_{RR}(x) = \Psi_R(x)$$

$$\rightarrow Fe^{ikx} + He^{-ikx} = Ce^{ikx} + De^{-ikx} \quad \left\{\text{@ } x = L\right\}$$

as we're checking for $x = L$

$$Fe^{ikL} + He^{-ikL} = Ce^{ikL} \longrightarrow (3)$$

and now,

$$\left.\frac{\partial \Psi_{RR}}{\partial x}\right|_{x=L} = \left.\frac{\partial \Psi_R}{\partial x}\right|_{x=L}$$

we'll have :

$$k_2\left(Fe^{ikL} - He^{-ikL}\right) = k_1 Ce^{ikL} \longrightarrow (4)$$

Hence, to explain the tunneling we need to define transmissibility factor ; which is $\left\{\begin{array}{l} \text{if (particle + energy} \rightarrow) \\ \text{if (particle + energy} \rightarrow) \end{array}\right\}$

as we know :

$$\Psi_R(x) = \underbrace{\Psi_{R_{right}}(x)}_{Ae^{ikx}} + \underbrace{\Psi_{R_{left}}}_{He^{-ikx}}$$

$$\Psi_L(x) = \Psi_{L_{right}}$$

so,

transmissibility should be :

for $\Psi_{R_{left}} \longrightarrow \Psi_{R_{left}}(x) = \Psi^*_{R_{left}}(x)$

$\hookrightarrow (Be^{-ikx})(Ae^{ikx}) \rightarrow A^2e^0 \rightarrow A^2$

for $\Psi_{R_{right}} \longrightarrow \Psi_{R_{right}}(x)\,\Psi^*_{R_{right}}(x)$

$\hookrightarrow (Ae^{ikx})(Ae^{-ikx}) \rightarrow A^2e^0 \rightarrow B^2$

for $\Psi_{L_{right}} \longrightarrow \Psi_{L_{right}}(x)\,\Psi^*_{L_{right}}(x)$

$\hookrightarrow (Ce^{ikx})(Ce^{-ikx}) \rightarrow C^2e^0 \rightarrow C^2$

* transmission coefficient (T) :

$$\frac{P_{L\,transmitted}}{P_{R_1\,incident}}$$

from above probabilities $\hookrightarrow$

$$\frac{C \times C}{A \times A} \rightarrow \left\{\begin{array}{l} t = \frac{C}{A} \\ T = t.t^* \end{array}\right\}$$

$$\Rightarrow \frac{C^2}{A^2} \longrightarrow (5)$$

* reflection coefficient (R) :

$$\frac{P_{R_1\,reflected}}{P_{R_1\,incident}} \rightarrow (6)$$

$$= \frac{B \times B}{A \times A} \rightarrow \left\{\begin{array}{l} r = \frac{B}{A} \\ R = r \times r \end{array}\right\}$$

now every thing is maths :')

divide A in eq 1

$$\frac{A + B}{A} = \frac{F + H}{A}$$

$$\frac{A}{A} + \frac{B}{A} = \frac{F}{A} + \frac{H}{A}$$

$$1 + r = f + g \quad \left\{\because \frac{F}{A} = f \;\&\; \frac{H}{A} = g\right\}$$

now, first divide every term of eq 2 with A ;

we'll have $\longrightarrow$ simplified eqn :

$$1 + r = f + g \longrightarrow \text{from (1)}$$

$$ik_1(1 - r) = k_2(f - g) \longrightarrow \text{from (2)}$$

$$fe^{ikL} + ge^{-ikL} = te^{ikL} \longrightarrow \text{from (3)}$$

$$k_2\left(fe^{ikL} - ge^{-ikL}\right) = ik_1te^{ikL} \longrightarrow \text{from (4)}$$

$$k_2\left(fe^{ikL} + ge^{-ikL}\right) = k_2te^{ikL}$$

$$\left(2k_2fe^{ikL} = ik_1te^{ikL} + k_2te^{ikL}\right)$$

$$f = \left(\frac{ik_1te^{ikL} + k_2te^{ikL}}{2k_2e^{ikL}}\right)$$

$$f = te^{ikL}\left(\frac{ik_1 + k_2}{2k_2e^{ikL}}\right)$$

$$f = e^{-ikL}\frac{te^{ikL}\left(ik_1 + k_2\right)}{2k_2}$$

$$f = te^{(ik_1 - ik_2)L}\left(\frac{ik_1 + k_2}{2k_2}\right)$$

as $f = \frac{F}{A}$ so,

$$\frac{F}{A} = te^{(ik_1 - ik_2)L}\left(\frac{ik_1 + k_2}{2k_2}\right)$$

now to get g value we'll substitute f in eq 1

$$\Rightarrow -2k_2ge^{-ikL} = ik_1te^{ikL} - k_2te^{ikL}$$

$$\Rightarrow g = \frac{te^{ikL}k_2 - ik_1te^{ikL}}{2k_2e^{-ikL}}$$

$$\hookrightarrow g = te^{(ik_2 + ik_1)L}\left(\frac{ik_2 - k_1}{-2k_2}\right)$$

to be continued $\longrightarrow$

hence,

$$\frac{\Omega}{A} = \frac{k\epsilon\left(\sin\alpha\sinh\beta\left(\beta k - \delta k\right)\right)}{-2k}$$

now, in eqn (i) do,

$$\beta k\,A + \beta k\,\delta = \beta k\,\xi + \beta k\,\eta$$

and add it with eqn (ii)

$$\beta k\,\xi + \beta k\,\eta = \delta k\,\xi + \delta k\,\eta$$

$$\beta k\,\xi - \beta k\,\eta = \delta k\,\xi - \delta k\,\eta$$

$$2\beta k + 0 = (\beta k + \delta k)\xi + (\beta k - \delta k)\eta$$

$$\hookrightarrow \quad 2\beta k = (\beta k + \delta k)\xi + (\beta k - \delta k)\eta$$

now grid plot these values of (ξ) & (η) that we plot found earlier

and we'll get :

$$2\beta k = \frac{(\beta k + \delta k)\,k\epsilon\left(\sin\alpha\sinh\beta\left(\beta k + \delta k\right)\right)}{2k} + \frac{(\beta k - \delta k)\,k\epsilon\left(\sin\alpha\sinh\beta\left(\beta k - \delta k\right)\right)}{-2k}$$

Yeah !! Yeah !! , I know it looks very bizarre and you wanna puke on it but

DON'T

its beautiful !!

simplifying it ;

$$2\beta k = k\epsilon\frac{\left(\sin\alpha\sinh\beta\left(\beta k + \delta k\right)\right)^2}{2k} + \frac{k\epsilon\left(\sin\alpha\sinh\beta\left(\beta k - \delta k\right)\right)^2}{-2k}$$

$$\Rightarrow \quad 2\beta k = \left[\frac{\epsilon\left(\sin\alpha\sinh\beta\left(\beta k + \delta k\right)\right)^2 - \epsilon\left(\sin\alpha\sinh\beta\left(\beta k - \delta k\right)\right)^2}{2k}\right]$$

$$\Rightarrow \quad 2\beta k = \left[\epsilon^{-2\beta k}\left(\beta k + \delta k\right)^2 - \epsilon^{2\beta k}\left(\beta k - \delta k\right)^2\right]\frac{k\epsilon\sin^2\alpha}{2k}$$

$$2\beta k = \left[\epsilon^{-2\beta k}\left(\beta k^2 - \delta k^2 + 2\beta k\delta k\right) - \epsilon^{2\beta k}\left(\beta k^2 - \delta k^2 - 2\beta k\delta k\right)\right]\frac{k\epsilon\sin^2\alpha}{2k}$$

let's look at above hyperbolic trig :

$$\left\{\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}, \quad \sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}\right\}$$

$$2\beta k = \left[\epsilon^{-2\beta k}\left(\delta k^2 - \beta k^2\right) + \epsilon^{2\beta k}\left(-2\left(\beta k\delta k\right)\right) - \epsilon^{2\beta k}\left(\delta k^2 - \beta k^2\right) + \epsilon^{2\beta k} + 2\left(\beta k\delta k\right)\right]\frac{k\epsilon\sin^2\alpha}{2k}$$

$$2\beta k = \left[\left(\beta k^2 - \delta k^2\right)\left(\epsilon^{-2\beta k} - \epsilon^{2\beta k}\right) + \left(\epsilon^{2\beta k} + \epsilon^{2\beta k}\right)\left(2\beta k\delta k\right)\right]\frac{k\epsilon\sin^2\alpha}{2k}$$

almost there, we'll get Jupyter Beautiful ... ❤

$\rightarrow$ we'll replace $(\epsilon^{-2\beta k} - \epsilon^{2\beta k})$ & $(\epsilon^{-2\beta k} + \epsilon^{2\beta k})$ with exponential form with higher bases functions !

$$-\sinh\left(2\beta k\right) = \frac{\epsilon^{-2\beta k} - \epsilon^{2\beta k}}{2} \quad \& \quad \cosh\left(2\beta k\right) = \frac{\epsilon^{-2\beta k} + \epsilon^{2\beta k}}{2}$$

$$2\beta k = \left[-\left(\beta k^2 + \delta k^2\right)2\sinh\left(2\beta k\right) + \left(2\beta k\delta k\right)2\cosh\left(2\beta k\right)\right]\frac{k\epsilon\sin^2\alpha}{2k}$$

finally we can solve for (ξ)

$$\xi = \frac{4\left(\beta k\delta k\right)\epsilon^{-2\beta k}}{\left[\left(\beta k^2 - \delta k^2\right)2\sinh\left(2\beta k\right) + 2\left(2\beta k\delta k\right)2\cosh\left(2\beta k\right)\right]}$$

$\rightarrow$ what we have here is ξ for T we must multiply it with its reciprocal complex

$$T = \left\{\frac{4\left(\beta k\delta k\right)\epsilon^{-2\beta k}}{\left[\left(\beta k^2 + \delta k^2\right)2\sinh\left(2\beta k\right) + 2\left(2\beta k\delta k\right)2\cosh\left(2\beta k\right)\right]}\right\} \times \left\{\frac{-4\left(\beta k\delta k\right)\epsilon^{2\beta k}}{\left[\left(\beta k^2 - \delta k^2\right)2\sinh\left(2\beta k\right) - 2\left(2\beta k\delta k\right)2\cosh\left(2\beta k\right)\right]}\right\}$$

don't be afraid pls, it's nothing spl or deep conjugate, I did it to get the sign of term having (i) in it

still to make it easy let me define some actions :

$$\rightarrow \quad \forall = \left(\beta k^2 - \delta k^2\right)$$

$$\S = \left(2\beta k\delta k\right)$$

now,

$$T = \frac{2i\,\S\epsilon^{2\beta k}}{\left\{\forall 2\sinh\left(2\beta k\right) + i\S 2\cosh\left(2\beta k\right)\right\}} \times \frac{-2i\,\S\epsilon^{-2\beta k}}{\left\{\forall 2\sinh\left(2\beta k\right) - i\S 2\cosh\left(2\beta k\right)\right\}}$$

$$\hookrightarrow \quad \left(\forall + i\S\right)\left(\forall - i\S\right) = \forall^2 + \S^2$$

$$\hookrightarrow \quad T = \frac{4\,\S^2}{4\,\forall^2\sinh^2\left(2\beta k\right) + 4\,\S^2\cosh^2\left(2\beta k\right)}$$

$$\Rightarrow \quad T = \frac{\S^2}{\forall^2\sinh^2\left(2\beta k\right) + \S^2\cosh^2\left(2\beta k\right)}$$

$$\left\{\cosh^2(x) - \sinh^2(x) = 1\right\}$$

hence,

$$T = \frac{\S^2}{\forall^2\sinh^2\left(2\beta k\right) + \S^2\left(1 + \sinh^2\left(2\beta k\right)\right)}$$

$$\Rightarrow \quad T = \frac{\S^2}{\forall^2\sinh^2\left(2\beta k\right) + \S^2 + \S^2\sinh^2\left(2\beta k\right)}$$

$$T = \frac{\S^2}{\S^2 + \left(\forall^2 + \S^2\right)\sinh^2\left(2\beta k\right)}$$

divide by $\S^2$

$$\hookrightarrow \quad T = \frac{1}{1 + \left(\frac{\forall^2}{\S^2} + 1\right)\sinh^2\left(2\beta k\right)}$$

now, just put all the value back in above eqn ! ❤

$\gamma = k_2 - k_1$

$\delta = 2k_2k_a$

$k_a = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$

$k_2 = \frac{\sqrt{2mE}}{\hbar}$

So, what is $\frac{\gamma^2}{\delta^2} + 1 = ?$

$\Rightarrow \frac{(k_2 - k_a)^2}{(2k_2k_a)^2} + 1$

$\Rightarrow \frac{k_2^2 + k_a^2 - 2k_2k_a}{4k_2^2k_a^2} + 1$

$\Rightarrow \frac{k_2^2 + k_a^2 - 2k_2k_a}{4k_2^2k_a^2} + \frac{4k_2^2k_a^2}{4k_2^2k_a^2}$

$\Rightarrow \frac{k_2^2 + k_a^2 + 2k_2k_a}{4k_2^2k_a^2}$

$\Rightarrow \frac{k_2^{\cancel{2}}}{4k_2^{\cancel{2}}k_a^2} + \frac{k_a^{\cancel{2}}}{4k_2^{\cancel{2}}k_a^{\cancel{2}}} + \frac{2k_2k_a}{4k_2^2k_a^2}$

$\Rightarrow \frac{k_2^2}{4k_a^2} + \frac{k_a^2}{4k_2^2} + \frac{1}{2}$

substitute !!! yes it's very very long !! 😅

$\Rightarrow \frac{\frac{2mE}{\hbar^2}}{4\left(\frac{2m(V_0 - E)}{\hbar^2}\right)} + \frac{\frac{2m(V_0 - E)}{\hbar^2}}{4\left(\frac{2mE}{\hbar^2}\right)} + \frac{1}{2}$

$\Rightarrow \frac{E}{4(V_0 - E)} + \frac{V_0 - E}{4E} + \frac{1}{2}$

$\Rightarrow \frac{E \times E}{4(V_0 - E) \times E} + \frac{(V_0 - E) \times (V_0 - E)}{4E(V_0 - E)} + 1 \quad \frac{E(V_0 - E)}{2E(V_0 - E)}$

$\Rightarrow \frac{E^2 + (V_0 - E)^2 + 2E(V_0 - E)}{4E(V_0 - E)}$

$\Rightarrow \frac{(E + (V_0 - E))^2}{4E(V_0 - E)} = \frac{V_0^2}{4E(V_0 - E)}$

so, $\frac{\gamma^2}{\delta^2} + 1 = \frac{V_0^2}{4E(V_0 - E)}$

T would be ;

$$T = \frac{1}{\frac{V_0^2}{4E(V_0 - E)} \sinh^2(k_a a) + 1}$$

$\{\eta = \frac{E}{V_0}\}$

therefore ;

$$T = \left(1 + \frac{\sinh^2(k_a a)}{4\eta_1(1 - \eta_1)}\right)^{-1}$$

and also we now can find reflection coefficient

as we know that a particle can either be transmitted or reflected

hence P(transmission) + P(reflection) = 1

$$T + R = 1$$

(R = 1 - T)

So plot these with $R < 1$ when $E < V_0$

$R > 1$ when $E > V_0$

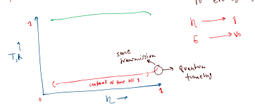
remember k_a is unitless

$k_a = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$

$k_a a = \frac{\sqrt{2mV_0(1 - \frac{E}{V_0})}}{\hbar}$

$k_a = \frac{\sqrt{2mV_0(1 - \eta)}}{\hbar}$

if we look at the graph



if pot. barrier is of 10^{-6} eV's of larger

$\eta_1 \to 1$

$E \to V_0$

a = 1 nm

similarly when we increase a the transmission factor shall be more certain of the penetration. depth of barrier energies comes closer for the height of the barrier

THANK YOU !!